ICTP Postgraduate Diploma Corse in Mathematics

Syllabus for Topology

September – October 2021

Instructor: Prof. Daniele Zuddas

Topological spaces. Topology on a set, topological spaces, open and closed subsets, subspaces, bases. Neighborhoods and neighborhood bases. Examples: trivial spaces, discrete spaces, R.

Metric spaces. Distance functions and metrizable spaces. Examples: discrete distance, Euclidean distance, R^n, B^n, S^n.

Topological operations. Union, product, metrizability of arbitrary unions and finite products. Example: the n-torus.

Continuous maps. Continuous maps between topological spaces and metric spaces, homeomorphisms, topological properties, continuity of the distance function.

Separation axioms. T_1, T_2 (Hausdorff spaces), T_3 (regular spaces), T_4 (normal spaces). Metrizability implies T_4. Urysohn's lemma (only for metrizable spaces).

Compactness. Open covers, compact spaces, embedding theorem for continuous injections from compact to Hausdorff spaces. Tychonoff's theorem (only for finite products). Compactness of $[0, 1]$ and the Heine-Borel theorem.

Quotient spaces. Equivalence relations on topological spaces, quotient topology, quotient space. Topology of real and complex projective spaces, compactness, Hausdorff property.

Topological operators. Closure, frontier, interior and exterior of subsets of topological spaces.

Connectedness. Connected spaces. Continuous image of a connected space. Connectedness of $[0, 1]$. Path-connected spaces, path-connected implies connected, continuous image of a path-connected space. Connected subspaces of R. Intermediate value theorem. Connectedness of a product space. Connected and locally path-connected implies path-connected. Connected and path-connected components. The standard example of a connected but not path-connected subspace of R^2.

Homotopy. Homotopic maps, relative homotopy, homotopy equivalence, contractible spaces, homotopy type as a topological property.

Fundamental group. Paths and loops, path multiplication, base point, fundamental group. Homomorphism induced by a continuous map, functoriality, homotopy invariance, topological invariance of the fundamental group. Retractions and strong deformation retractions. Triviality of the fundamental group of a contractible space. Independence of base point. Fundamental group of a product space.

Seifert-Van Kampen theorem. Free groups, free product of groups, group presentations. Statement of the Seifert-Van Kampen theorem (without proof), presentation of the fundamental group.

Covering spaces. Definition of coverings, examples. Lifting properties for paths and homotopies of paths. Simply-connected spaces; S^n is simply-connected for $n \geq 2$. Fundamental group of S^1. Characterization of simply-connected spaces in terms of paths. Isomorphism of coverings. Universal coverings: unicity (with proof), existence (without proof). Universal covering of the n-torus. No-retraction theorem and Brouwer's fixed-point theorem.

Group actions. Group actions on topological spaces. Effective, free and properly discontinuous actions. Regular coverings. Properly discontinuous actions on simply-connected spaces and fundamental group. Fundamental groups of real projective spaces.

Computing fundamental groups. Bouquet of circles, quotients of a regular polygon and fundamental groups of compact orientable surfaces (outline).

References

1) C. Kosniowski, *A first course in algebraic topology*, Cambridge University Press, 1980.

2) I. Singer and J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Springer-Verlag, 1967.

3) J. R. Munkres, *Topology*, Pearson, 2000.